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RENEWAL THEOREMS FOR MARKOV CHAINS

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1. Introduction

This study originated in an attempt to understand the meaning of two parameters, the mean and the variance, which appear in the limit theorems concerning sums of identically distributed independent random variables. The law of large numbers, when written in the usual form ($\lim S_n/n = \mu$) does not immediately suggest a natural generalization to Markov chains; the reason is that a Markov chain takes its values in an abstract state space so that not only the limit μ, but also the ratios which tend to μ have to be reinterpreted in a meaningful fashion. Therefore, we proceed to the renewal theorem in the form proved by Erdös, Feller, and Pollard (cf. [3], p. 286) and Chung and Wolfowitz [2], and formulate it in a manner which readily suggests a natural generalization.

Consider a random walk (spatially homogeneous Markov chain) x_n on the integers with transition function $P(x, y)$ defined for arbitrary pairs of integers x, y. Suppose that it satisfies

$$P(x, y) \geq 0, \qquad \sum_{y=-\infty}^{\infty} P(x, y) = 1, \tag{1.1}$$

$$P(x, y) = P(x + z, y + z) \qquad \text{for all} \quad z, \tag{1.2}$$

$$\sum_{x=-\infty}^{\infty} |x|\, P(0, x) < \infty, \qquad \sum_{x=-\infty}^{\infty} xP(0, x) = \mu > 0, \tag{1.3}$$

$$\sum_{y=-\infty}^{\infty} P(x, y)f(y) = f(x) \quad \text{and} \quad |f(x)| \leq 1 \Rightarrow f(x) = \text{constant}. \tag{1.4}$$

The last condition (1.4) is well known to be equivalent to the usual aperiodicity requirement that the support of $P(0, x)$ is not contained in a proper subgroup of the integers (cf. [4], p. 276). This Markov chain is transient in view of condition (1.3), ([4], p. 33), and the renewal theorem is a simple statement concerning the asymptotic behavior of the Green function $G(x, y)$ defined by

$$\begin{aligned} G(x, y) &= \sum_{n=0}^{\infty} P_n(x, y), \\ P_0(x, y) &= \delta(x, y), \\ P_{n+1}(x, y) &= \sum_{z=-\infty}^{\infty} P_n(x, z)P(z, y), \qquad n \geq 0. \end{aligned} \tag{1.5}$$

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The renewal theorem states that conditions (1.1) through (1.4) imply

$$\lim_{y \to +\infty} G(x, y) = \frac{1}{\mu}. \tag{1.6}$$

The mean μ measures, in some sense, the rate of drift of the Markov chain to its boundary. To see this phenomenon in its simplest setting, suppose in addition to (1.1)–(1.4) that the random walk can only go to the right, namely that

$$P(x, y) = 0 \quad \text{when} \quad y \leq x, \tag{1.7}$$

and let

$$T_n = \min [k | x_k > n]. \tag{1.8}$$

Then a weak (Césaro) version of (1.6) may be written in the form

$$\sum_{x=0}^{n} G(0, x) = E_0[T_n] \sim \frac{n}{\mu} \qquad \text{as} \quad n \to \infty. \tag{1.9}$$

Consider now a Markov chain x_n whose state space consists of the integers, and whose transition function $P(x, y)$ satisfies (1.1) and (1.7), but neither (1.2), (1.3), nor (1.4). Its Green function $G(x, y)$, defined by (1.5), will obviously be finite, and the identity in (1.9) is still correct. The problem of understanding the role of μ may therefore be phrased as follows: to find a function f, defined in a natural manner, so that

$$\sum_{x=0}^{n} G(0, x) = E_0[T_n] \sim f(n) \qquad \text{as} \quad n \to \infty. \tag{1.10}$$

We shall proceed to show that this problem may be solved, for a large class of Markov chains x_n, with countable state space S, by choosing for f a solution of the equation

$$\sum_{y \in S} P(x, y) f(y) - f(x) = 1, \qquad \sum_{y \in S} P(x, y) |f(y)| < \infty, \qquad x \in S. \tag{1.11}$$

(We shall simply write $Pf - f = 1$, in the sequel, to indicate that f satisfies the equation in (1.11) and is summable with respect to P.) In the case of random walk satisfying (1.1), (1.2), and (1.3), the equation $Pf - f = 1$ is indeed satisfied by the function

$$f(n) = \frac{n}{\mu}, \qquad n = 0, \pm 1, \pm 2, \cdots, \tag{1.12}$$

as an elementary calculation shows. It is perhaps more remarkable that even the variance of random walk is closely related to equation (1.11). For suppose that (1.1) and (1.2) hold, but that (1.3) is replaced by

$$\sum_{x=-\infty}^{\infty} x P(0, x) = \mu = 0, \qquad 0 < \sigma^2 = \sum_{x=-\infty}^{\infty} x^2 P(0, x) < \infty. \tag{1.13}$$

Then (1.12) is meaningless, while

$$f(n) = \frac{n^2}{\sigma^2}, \qquad n = 0, \pm 1, \pm 2, \cdots \tag{1.14}$$

now becomes a solution of $Pf - f = 1$.

In section 2 we show how certain solutions of (1.11) lead to a law of large numbers for Markov chains which generalizes the usual law of large numbers. In section 3 we give sufficient conditions for the weak renewal theorem (1.10) to hold. Finally, section 4 is devoted to a renewal theorem for a class of Markov chains which generalizes the renewal theorem (1.6) for random walk with finite second moment. It is the only satisfactory result of this paper, in the sense that it takes the form of a uniqueness theorem for solutions of equation (1.11) satisfying certain growth conditions. And this is, of course, the heart of the matter: we have to learn how to single out that solution of (1.11) which is the natural analogue of the linear function in (1.12), and this causes grave difficulties.

It is easy to construct Markov chains for which (1.11) has no solution at all. On the other hand, there may obviously be many solutions, since $f + h$ satisfies (1.11) whenever f does and h is harmonic, that is, satisfies $Ph = h$. But while boundary theory characterizes the nonnegative harmonic functions, nothing is known about the set of all harmonic functions.

Finally, it may even happen that (1.11) has one, or even more than one, solution, but that none of them are sufficiently well behaved to yield even a weak limit theorem of the type of (1.9).

2. A law of large numbers

Let us assume that x_n is a Markov chain, whose state space is a countable set S, and whose transition function P satisfies (1.1) (with the summation over S, of course). A martingale convergence theorem due to Chow, Mallows, and Robbins [1] states that if ξ_n is a centered sequence of random variables (the conditional expectation of ξ_{n+1} given $\xi_1, \cdots, \xi_n$ is 0) and if $\sum E|\xi_n|^\alpha < \infty$ for some $1 \le \alpha \le 2$, then the series $\sum \xi_n$ converges with probability one. As an immediate consequence, we can obtain a strong law of large numbers for x_n.

THEOREM 2.1. *Suppose that the transition function P of x_n is such that the equation $Pf - f = 1$ has a solution f with the property that*

$$\sum_{y \in S} P(x, y)|f(y) - f(x) - 1|^\alpha < M, \qquad \text{for all} \quad x \in S, \tag{2.1}$$

for some $\alpha > 1$ and some $M < \infty$. Then

$$P_x\left[\lim_{n\to\infty} \frac{f(x_n)}{n} = 1\right] = 1 \qquad \text{for each} \quad x \in S. \tag{2.2}$$

PROOF. We write

$$\begin{aligned} &f(x_n) - f(x_0) - n \\ &\qquad = \sum_{k=0}^{n-1} y_k, \qquad y_k = f(x_{k+1}) - f(x_k) - 1, \qquad k \ge 0, \qquad n \ge 1, \end{aligned} \tag{2.3}$$

and observe that in view of $Pf - f = 1$, the sequence $f(x_n) - n$, for $n \ge 0$, is a martingale with respect to the fields $\mathfrak{F}_n = \{x_0, x_1, \cdots, x_n\}$, $n \ge 0$.

For the same reason, the sequence

$$z_n = \sum_{k=1}^{n-1} \frac{y_k}{k}, \qquad n \geq 1 \tag{2.4}$$

is also a martingale with respect to the same fields. By use of Kronecker's lemma, one can obtain (2.2) if one knows that the limit of z_n exists and is finite with probability one with respect to the measure $P_x[\cdot]$ for each x (in other words, for each starting point $x_0 = x$ of the Markov chain). But in view of the theorem of Chow, Mallows, and Robbins, the series in (2.4) will converge with probability one for each x if

$$\sum_{k=0}^{\infty} E_x \left|\frac{y_k}{k}\right|^{\alpha} < \infty \qquad \text{for each} \quad x \in S. \tag{2.5}$$

Now (2.1) implies that

$$\begin{aligned} E_x|y_k|^{\alpha} &= E_x|f(x_{k+1}) - f(x_k) - 1|^{\alpha} \\ &= \sum_{y \in S} P_x[x_k = y] \sum_{z \in S} P(y, z)|f(z) - f(y) - 1|^{\alpha} \leq M, \end{aligned} \tag{2.6}$$

so that (2.5) holds, and the theorem is proved.

There are examples which show that the conclusion of the theorem is false when (2.1) only holds with $\alpha = 1$. An interesting open question concerns the Green function of a Markov chain satisfying the hypotheses of the theorem. It follows from (2.2) that the process must be transient so that $G(x, y) < \infty$ for all x and y in S, but it is not clear whether the existence of a function f satisfying (2.1) implies that $G(x, x)$ must be bounded.

It is, of course, possible to rephrase the theorem, by making sufficiently strong assumptions concerning the Green function of the process x_n, so that the hypotheses of the theorem become conclusions.

THEOREM 2.2. *Suppose that the Markov chain is transient, and that its Green function has the property that*

$$h(x, y) = \sum_{z \in S} |G(x, z) - G(y, z)| < \infty, \qquad x, y \in S, \tag{2.7}$$

and that

$$\sum_{y \in S} P(x, y)|h(x, y)|^{\alpha} \leq M, \qquad x \in S, \alpha > 1. \tag{2.8}$$

Then, for any fixed point $0 \in S$,

$$f(x) = \sum_{z \in S} [G(0, z) - G(x, z)] \tag{2.9}$$

is a solution of $Pf - f = 1$, *and the strong law* (2.2) *holds.*

PROOF. It follows from (2.8) that the function f defined by (2.9) is summable with respect to P. Also (2.7), together with (2.8), yields the dominated convergence necessary to interchange limits in

$$\begin{aligned} Pf(x) - f(x) &= \sum_{z \in S} \sum_{y \in S} P(x, y)[G(0, z) - G(y, z)] - f(x) \\ &= \sum_{z \in S} [G(0, z) - G(x, z) + \delta(x, z)] - f(x) = 1. \end{aligned} \tag{2.10}$$

Now (2.8) and (2.9) imply (2.1), so that the strong law follows from the previous theorem.

3. A weak renewal theorem

Let us investigate conditions on a Markov chain x_n, under which the stopping times T_n defined in (1.8) have the property that

$$\lim_{n\to\infty} \frac{E_n[T_n]}{f(n)} = 1 \tag{3.1}$$

for some solution f of the equation $Pf - f = 1$. For (3.1) to make sense it is necessary that the state space S be ordered. Therefore, we may take for S the positive integers, but this assumption is of course quite vacuous unless we assume that the transition function P is somehow well behaved with respect to this particular ordering. This motivates the following hypotheses (A) (which can be weakened in some respects, but at the cost of complicating the analysis without leading to really satisfactory results).

(A) The Markov chain x_ν, $\nu \geq 0$, has as its state space the positive integers. Its transition function $P(k, \ell)$has the boundedness property that the chain can take only a finite number of steps to left or right: there is an $M > 0$ such that $P(k, \ell) = 0$ when $|\ell - k| > M$. Finally, it is assumed that the chain has no finite closed set of states: there is no finite subset C of the integers such that $\sum_{\ell\in C} P(k, \ell) = 1$ for all $k \in C$.

Note that hypothesis (A) does not imply that the chain x_ν is transient. Thus the Green function may be infinite, but the stopping times T_n in (3.1) are finite with probability one and have finite expectation. (This follows from our assumption that there are no closed sets; therefore, the first time T_n at which the chain x_ν is outside the set $[1, 2, \cdots, n]$ has the property that the tails $P_x[T_n > m]$ decrease at least exponentially as $m \to \infty$, for each fixed x and n.)

THEOREM 3.1. *Suppose that the Markov chain x_ν, $\nu \geq 0$, satisfies (A), and let $T_n = \min [\nu | \nu \geq 0, x_\nu > n]$. Suppose further that $Pf - f = 1$ has a solution f such that $f(n + 1)/f(n) \to 1$ as $n \to +\infty$. Then*

$$\lim_{n\to\infty} \frac{E_k[T_n]}{f(n)} = 1, \qquad \text{for each} \quad k \geq 1. \tag{3.2}$$

PROOF. We begin by showing that any solution of $Pf - f = 1$ such that $f(n + 1)/f(n) \to 1$ has the property

$$\lim_{n\to\infty} f(n) = +\infty. \tag{3.3}$$

Suppose that f is such a solution, and let

$$m_n = \max_{1\leq k\leq n} f(k). \tag{3.4}$$

It is impossible that $f(k) = m_n$ when $k \leq n - M$, since $k \leq n - M$ implies

$$f(k) = \sum_{\ell=k-M}^{k+M} P(k, \ell)f(\ell) - 1 \leq m_n - 1. \tag{3.5}$$

Therefore, f assumes its maximum in $[1, 2, \cdots, n]$ at one of the points of the interval $[n - M + 1, n]$. But then the identity in (3.5) implies that $f(k) \geq m_n + 1$ at one of the points k in the interval $[n + 1, n + M]$. Now we may repeat the above argument, with n replaced by $n + M$ and obtain the following conclusion: there exists an increasing sequence of integers $n_1 < n_2 < n_3 < \cdots$ such that

$$n_{k+1} - n_k \leq M \quad \text{and} \quad f(n_{k+1}) - f(n_k) \geq 1. \tag{3.6}$$

Thus $f(n_k) \to +\infty$ as $k \to \infty$, and (3.6), in conjunction with the hypothesis that $f(n + 1)/f(n) \to 1$, implies (3.3).

The rest of the proof depends on the identity

$$E_k[f(x_{T_N})] - E_k[T_n] = f(k), \qquad k \geq 1, n \geq 1. \tag{3.7}$$

(It should be clear how (3.7) implies the theorem; simply divide (3.7) by $f(n)$ and let $n \to \infty$, observing that $f(x_{T_N})/f(n) \to 1$ boundedly with probability one, since $|x_{T_N} - n| \leq M$.)

Equation (3.7) is suggested by the fact, observed in the proof of theorem 3.1, that $f(x_n) - n$ is a martingale. Thus for every bounded stopping time T, $E_k[f(x_T) - T] = f(k)$. But since the stopping times T_n are not bounded, we have to use the slightly stronger Markov property instead.

Let C be an arbitrary finite set of positive integers and $T = \min [\nu | \nu \geq 0, x_\nu \in \overline{C}]$, where $\overline{C}$ is the complement of C. For each integer x,

$$\begin{aligned} E_x[f(x_T)] &= \lim_{m\to\infty} \sum_{\nu=0}^{m} \sum_{z\in C} P_x[x_\nu = z; T > \nu] \sum_{y\in \overline{C}} P(z, y) f(y) \\ &= \lim_{m\to\infty} \sum_{\nu=0}^{m} \sum_{z\in C} P_x[x_\nu = z; T > \nu]\{f(z) + 1 - \sum_{y\in C} P(z, y) f(y)\} \\ &= \lim_{m\to\infty} \left\{\sum_{\nu=0}^{m} \sum_{z\in C} P_x[x_\nu = z; T > \nu][f(z) + 1] \right. \\ &\qquad \left. - \sum_{\nu=1}^{m+1} \sum_{z\in C} P_x[x_\nu = z; T > \nu] f(z)\right\} \\ &= \lim_{m\to\infty} \left\{\sum_{\nu=0}^{m} P_x[T > \nu] + f(x) - \sum_{z\in C} P_x[x_\nu = z; T > m + 1] f(z)\right\}. \end{aligned} \tag{3.8}$$

Hence,

$$E_x[f(x_T)] - E_x[T] - f(x) = \lim_{m\to\infty} \sum_{z\in C} P_x[x_\nu = z; T > m + 1] f(z). \tag{3.9}$$

The right-hand limit is 0 since, if $|C|$ denotes the cardinality of C,

$$\Big|\sum_{z\in C} P_x[x_\nu = z, T > m + 1] f(z)\Big| \leq |C| \max_{x\in C} |f(x)| P_x[T > m + 1] \to 0. \tag{3.10}$$

Thus (3.7) is proved by setting $T = T_n$, and that completes the proof of the theorem.

Just as was done in section 2, one can modify theorem 3.1 so that the hypotheses concerning f become conclusions instead. Suppose that a process x_ν satisfies (A) and that in addition $G(x, y) < \infty$, and

$$f(x) = \sum_{y=1}^{\infty} [G(1, y) - G(x, y)] < \infty \qquad \text{for each} \quad x, \tag{3.11}$$

and

$$\lim_{x \to +\infty} \frac{f(x+1)}{f(x)} = 1. \tag{3.12}$$

Then it follows that the function f defined by (3.11) satisfies $Pf - f = 1$. Therefore, the conclusion of theorem 3.1 is valid and takes on the curious form

$$\lim_{n\to\infty} \frac{1}{E_k[T_n]} \sum_{y=1}^{\infty} [G(1, y) - G(n, y)] \tag{3.13}$$

$$= \lim_{n\to\infty} \frac{1}{E_k[T_n]} \lim_{m\to\infty} \{E_1[T_m] - E_n[T_m]\} = 1, \qquad k \geq 1.$$

As an illustration of theorem 3.1 in the recurrent case, consider a random walk on the integers (positive and negative) satisfying (1.1), (1.2), and (1.3) with mean $\mu = 0$. Assume also that the possible displacements are bounded. Then $\sigma^2 < \infty$, and we shall be concerned with the solution f of $Pf - f = 1$, given in equation (1.13). It is natural to order the state space in such a way that f becomes monotone. Therefore, we map the integers on the positive integers according to the formula $k \to 2k$ for $k \geq 0$ and $k \to 2|k| - 1$ when $k < 0$. Thus the random walk induces a Markov chain on $[1, 2, 3, \cdots)$ for which $Pf - f = 1$ has the solution

$$f(j) = \begin{cases} j^2/4\sigma^2 & \text{for even} \quad j \geq 1, \\ (j+1)^2/4\sigma^2 & \text{for odd} \quad j \geq 1. \end{cases} \tag{3.14}$$

The conditions of theorem 3.1 are satisfied, T_n becomes the time elapsed until the random walk on the group of all integers leaves an interval which is approximately $[-n/2, n/2]$, and theorem 3.1 yields the well-known conclusion $E_k[T_n] \sim n^2/\sigma^2$ as $n \to \infty$, for each fixed k.

4. A renewal theorem

Here we make drastically stronger assumptions than in section 3, in order to arrive at a generalization of the classical renewal theorem for positive random variables. We consider processes x_ν, $\nu \geq 0$, satisfying conditions

(B) (i) *x_ν, $\nu \geq 0$ is a Markov chain on the positive integers, which can move only to the right; that is, $P(k, \ell) = 0$ when $\ell \leq k$;*

(ii) *the chain has uniformly integrable second moments, namely, the tails are assumed uniformly bounded so that*

$$\tau(n) = \sup_{1 \leq k < \infty} \sum_{j=n+k+1}^{\infty} P(k, j) < \infty; \tag{4.1}$$

and in addition $\sum_1^{\infty} n\tau(n) < \infty$;

(iii) *the only bounded harmonic functions are the constants, that is, $Pf = f$, and $|f| \leq M \Rightarrow f = \text{constant}$.*

The renewal theorem takes the form

THEOREM 4.1. *If a Markov chain satisfies* (B) *and if the equation* $Pf - f = 1$ *has a solution* f *whose increments are bounded* ($|f(n+1) - f(n)| \leq M$ *for all* $n \geq 1$), *then*

$$f(n) - f(k) = \sum_{\ell=1}^{\infty} [G(k, \ell) - G(n, \ell)] \qquad \text{for all} \quad k \geq 1, n \geq 1. \tag{4.2}$$

Before proceeding to the proof, let us observe how this uniqueness theorem for a class of solutions of $Pf - f = 1$ reduces to the ordinary renewal theorem for random walk.

COROLLARY 4.1. *Let* $P(k, \ell)$ *be the transition function of a positive aperiodic random walk on the positive integers with finite second moment, that is, assume that*

$$P(k, \ell) \geq 0, \qquad \sum_{\ell=1}^{\infty} P(k, \ell) = 1, \qquad \sum_{\ell=1}^{\infty} \ell^2 P(k, \ell) < \infty, \tag{4.3}$$
$$P(k, \ell) = P(1, \ell - k + 1),$$

and that the greatest common divisor of $[k|P(1, 1 + k) > 0]$ *is* 1. *Let* $G(k, \ell)$ *be the Green function defined in* (1.5). *Then*

$$\lim_{n\to\infty} G(1, n) = \frac{1}{\mu} \quad \text{where} \quad \mu = \sum_{k=1}^{\infty} kP(1, 1 + k). \tag{4.4}$$

PROOF OF THE COROLLARY. All the hypotheses for theorem 4.1 are clearly satisfied if we choose $f(n) = n/\mu$, $n \geq 1$. Now set $n = 2$ and $k = 1$ in equation (4.2). It reduces to

$$\begin{aligned} \frac{1}{\mu} &= \lim_{m\to\infty} \sum_{\ell=1}^{m} [G(1, \ell) - G(2, \ell)] \\ &= \lim_{m\to\infty} \sum_{\ell=1}^{m} [G(1, \ell) - G(1, \ell - 1)] = \lim_{m\to\infty} G(1, m). \end{aligned} \tag{4.5}$$

PROOF OF THE THEOREM. First we need the fact that

$$\lim_{n\to\infty} [G(k, n) - G(\ell, n)] = 0 \tag{4.6}$$

for every pair of positive integers k and ℓ. This is a standard ingredient of every potential theoretical proof of the renewal theorem. If (4.6) were false for some pair k, ℓ, with $k \neq \ell$, then there would exist a subsequence n' of the positive integers such that the limit in (4.6) along this subsequence is $\alpha \neq 0$. (Here and in the sequel we use the fact that $0 \leq G(k, \ell) \leq 1$, in view of condition B(i).) Taking a further subsequence n'' of n' with the property that $\lim_{n''\to\infty} G(k, n'') = \phi(k)$ exists for all $k \geq 1$, we find that $P\phi = \phi$, namely that ϕ is a bounded harmonic function. However, $\phi(k) - \phi(\ell) = \alpha \neq 0$, which shows that ϕ is not constant. That contradicts assumption B(iii) and therefore proves (4.6).

Now define, as usual $T_m = \min [\nu|\nu \geq 0, x_\nu > m]$, observe that in view of B(i),

$$\sum_{\ell=1}^{m} G(k, \ell) = E_k[T_m], \tag{4.7}$$

and that

$$E_k[f(x_{T_m})] = E_k[T_m] + f(k), \qquad k \geq 1, \qquad m \geq 1. \tag{4.8}$$

(The proof of (4.8) as given in section 3 depends on B(i); neither B(ii) nor B(iii) are used.) Combining (4.7) and (4.8), we obtain

$$\begin{aligned} \sum_{\ell=1}^{m} [G(1, \ell) - G(n, \ell)] - [f(n) - f(1)] \\ &= E_1[f(x_{T_m})] - E_n[f(x_{T_m})] \\ &= E_1[f(x_{T_m}) - f(m)] - E_n[f(x_{T_m}) - f(m)]. \end{aligned} \tag{4.9}$$

Decomposing these expectations according to the place of the last visit of the process to the interval $[1, m]$, one obtains

$$\begin{aligned} \sum_{\ell=1}^{m} [G(1, \ell) - G(n, \ell)] - [f(n) - f(1)] \\ = \sum_{j=1}^{m} [G(1, j) - G(n, j)] \sum_{k=1}^{\infty} P(j, m + k)[f(m + k) - f(m)]. \end{aligned} \tag{4.10}$$

Now set $|G(1, j) - G(n, j)| = \epsilon_j$ and observe that

$$|f(m + k) - f(m)| \leq kM, \qquad k \geq 1. \tag{4.11}$$

Then (4.10) gives

$$\begin{aligned} \left| \sum_{\ell=1}^{m} [G(1, \ell) - G(n, \ell)] - [f(n) - f(1)] \right| \\ \leq M \sum_{j=1}^{m} \epsilon_j \sum_{k=1}^{\infty} kP(j, m + k). \end{aligned} \tag{4.12}$$

By condition B(ii),

$$\begin{aligned} \sum_{k=1}^{\infty} kP(j, m + k) \\ = \sum_{k=1}^{\infty} \sum_{r=k}^{\infty} P(j, j + m - j + r) \leq \sum_{k=1}^{\infty} \tau(m - j + k - 1), \end{aligned} \tag{4.13}$$

so that

$$\begin{aligned} M \sum_{j=1}^{m} \epsilon_j \sum_{k=1}^{\infty} kP(j, m + k) &\leq M \sum_{j=1}^{m} \epsilon_j \sum_{k=0}^{\infty} \tau(m - j + k) \\ &= M \sum_{j=1}^{m} \epsilon_{m-j}\delta_j, \qquad \text{where} \quad \delta_j = \sum_{k=j}^{\infty} \tau(k). \end{aligned} \tag{4.14}$$

By B(ii) $\sum_1^\infty \delta_j < \infty$, and by (4.6), $\epsilon_j \to 0$. Therefore,

$$\lim_{m \to \infty} \sum_{j=1}^{m} \epsilon_{m-j}\delta_j = 0, \tag{4.15}$$

which implies, according to (4.12) and (4.13), that

$$\lim_{m \to \infty} \sum_{\ell=1}^{m} [G(1, \ell) - G(n, \ell)] = f(n) - f(1). \tag{4.16}$$

Thus we have proved (4.2) in the case when $k = 1$, but this special case of (4.2) is easily seen to imply the general case. That completes the proof.

Many open problems remain, however. If one tries to dispense with, or weaken, condition B(i), then it is hard to derive (4.6) as the Green function $G(k, \ell)$ need apparently no longer be bounded. Keeping B(i) and B(iii), however, it seems certain that theorem 4.1 should hold under a weaker version of B(ii). It is not known whether it suffices to assume that the tails are uniformly summable, namely that $\sum \tau(n) < \infty$.

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